# Berry phase for a potential well transported in a homogeneous magnetic field

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**Abstract**

We consider a two-dimensional particle of charge $e$ interacting with a homogeneous magnetic field perpendicular to the plane and a potential well which is transported along a closed loop in the plane. We show that a bound state corresponding to a simple isolated eigenvalue acquires at that Berry's phase equal to $2\pi \operatorname{sgn} e$ times the number of flux quanta through the oriented area encircled by the loop. We also argue that this is a purely quantum effect since the corresponding Hannay angle is zero.

There are many different situations in which a nontrivial Berry phase [Be1] is observed. For instance, the effect was studied recently in mesoscopic systems [LSG, MHK] for particles with spin interacting with a time-dependent magnetic field. However, a uniform magnetic field can give rise to a Berry phase even if the spin-orbital coupling is neglected. An example has been found in the paper [EG] within a model which describes a charged particle confined by a point interaction and placed into a magnetic field of constant direction, which is independent of time and may be homogeneous; the phase emerges when the $\delta$ potential (understood in the sense of [AGHH]) moves along a closed loop $\mathcal{C}$ in the plane.

The aim of this letter is to demonstrate that the same is true when the point interaction is replaced by any potential capable of binding the particle, and moreover, that the Berry phase is in this situation again given explicitly as a multiple of the flux through the area encircled by $\mathcal{C}$. The trick we use is adopted from another example discussed in [EG] being based on the observation that the transport of the potential well can be equivalently expressed by means of the magnetic translation group [Za].

We consider therefore a two-dimensional spinless quantum particle described by the Hamiltonian

$$H = H^0 + V(\vec{r})\,, \tag{1}$$

where

$$H^0 = -\frac{\hbar^2}{2m}\left[\,(-\partial_x + \pi i\xi y)^2 + (-\partial_y - \pi i\xi x)^2\,\right] \tag{2}$$

is the free Schrödinger operator with the magnetic field in the circular gauge,

$$\vec{A} = \frac{1}{2}\,\vec{B}\times\vec{r}, \tag{3}$$

where $\vec{B} = B\vec{e}_3$ is the field strength and the quantity $\xi$ appearing in (2) is the magnetic flux density,

$$\xi = \frac{B}{\Phi_0}\,\mathrm{sgn}\,e \tag{4}$$

with $\Phi_0 = 2\pi\hbar c/|e|$ being the flux quantum. For the sake of simplicity we employ in the following the rational system of units, $|e| = \hbar = c = 2m = 1$.

We suppose that the potential $V$ is such that the operator (1) is self-adjoint and has at least one simple eigenvalue which we call $E_0$; the corresponding normalized eigenfunction will be denoted as $\psi_0$. We will consider the family of shifted operators

$$H(\vec{a}) = H^0 + V(\vec{r}-\vec{a}) \tag{5}$$

and denote by $V_{\vec{a}}$ the operator of multiplication by $V(\vec{r}-\vec{a})$. Let $[\vec{a}]$ be the operator of magnetic translation on the vector $\vec{a}$,

$$[\vec{a}]f(\vec{r}) = \exp(-\pi i\xi\vec{r}\wedge\vec{a})\,f(\vec{r}-\vec{a})\,, \tag{6}$$

then the intertwining relation

$$[\vec{a}]V_0 = V_{\vec{a}}[\vec{a}] \tag{7}$$

is valid. It follows that $\psi_{\vec{a}}(\vec{r}) = [\vec{a}]\psi_0(\vec{r})$ is an eigenfunction of the operator $H(\vec{a})$ corresponding to the eigenvalue $E_0$.

We want to find the Berry phase which refers to an adiabatic evolution of the system with the Hamiltonian $H(\vec{a})$ when the vector $\vec{a}$ which characterizes the potential position moves along a loop $\mathcal{C}$. To this end we have to find the corresponding Berry potential

$$U(\vec{a}) = i\langle\psi_{\vec{a}}|\nabla_{\vec{a}}\psi_{\vec{a}}\rangle \, . \tag{8}$$

From (6) we get

$$\begin{aligned}
\partial_{a_1}\psi_{\vec{a}}(x,y) &= \pi i\xi y\psi_{\vec{a}}(x,y) + \exp(-\pi i\xi\vec{r}\wedge\vec{a})\,\partial_{a_1}\psi_0(x-a_1,y-a_2) \\
&= \pi i\xi y\psi_{\vec{a}}(x,y) - \exp(-\pi i\xi\vec{r}\wedge\vec{a})\,\partial_{x}\psi_0(x-a_1,y-a_2)
\end{aligned} \tag{9}$$

and

$$\partial_{a_2}\psi_{\vec{a}}(x,y) = -\pi i\xi x\psi_{\vec{a}}(x,y) - \exp(-\pi i\xi\vec{r}\wedge\vec{a})\,\partial_{y}\psi_0(x-a_1,y-a_2)\, .$$

Consequently,

$$\begin{aligned}
\langle\psi_{\vec{a}}|\partial_{a_1}\psi_{\vec{a}}\rangle &= \iint \pi i\xi(y-a_2)|\psi_0(x-a_1,y-a_2)|^2 dx\,dy \\
&+ \pi i\xi a_2\iint |\psi_0(x,y)|^2 dx\,dy \\
&- \iint \overline{\psi_0(x-a_1,y-a_2)}\,\partial_x\psi_0(x-a_1,y-a_2)\,dx\,dy \\
&= \pi i\xi a_2 + \pi i\xi c_1\, ,
\end{aligned} \tag{10}$$

where

$$c_1 = \iint y\,|\psi_0(x,y)|^2 dx\,dy - \frac{1}{\pi i\xi}\iint \overline{\psi_0(x,y)}\,\partial_x\psi_0(x,y)\,dx\,dy \tag{11}$$

is independent of $\vec{a}$. Analogously,

$$\langle\psi_{\vec{a}}|\partial_{a_2}\psi_{\vec{a}}\rangle = -\pi i\xi a_1 + \pi i\xi c_2 \tag{12}$$

with

$$c_2 = \iint x\,|\psi_0(x,y)|^2 dx\,dy - \frac{1}{\pi i\xi}\iint \overline{\psi_0(x,y)}\,\partial_y\psi_0(x,y)\,dx\,dy\, . \tag{13}$$

This yields

$$U_1(\vec{a}) = -\pi\xi a_2 - \pi\xi c_1\,, \quad U_2(\vec{a}) = \pi\xi a_1 - \pi\xi c_2\,.$$

Moreover, the second terms can be removed by an appropriate gauge transformation, so we get

$$U(\vec{a}) = \pi\xi(-a_2, a_1)\,; \tag{14}$$

notice that the expression coincides with the vector potential of the homogeneous magnetic field. Using (14) we find that the corresponding Berry phase is given by

$$\gamma(\mathcal{C}) = \pi\xi \int_{\mathcal{C}} (a_1 da_2 - a_2 da_1) = 2\pi\xi S\,, \tag{15}$$

where $S$ is the (oriented) area encircled by the loop $\mathcal{C}$, in other words,

$$\gamma(\mathcal{C}) = 2\pi\,\mathrm{sgn}\,e\,\frac{\Phi_{\mathcal{C}}}{\Phi_0}\,, \tag{16}$$

where $\Phi_{\mathcal{C}}/\Phi_0$ is the number of flux quanta through $S$. This is the result we have announced in the opening.

Let us add a comment on the meaning of the result. Consider a classical dynamical system with the Hamiltonian $H^0 + V(\vec{r})$ which is completely integrable. Recall that the Hannay angle $\Delta\theta(\mathcal{C})$ can be defined [Han] for a cyclic adiabatic evolution of the system with the "shifted" Hamiltonian $H(\vec{a}) = H^0 + V(\vec{r}-\vec{a})$ along a closed loop $\mathcal{C}$ is the parameter space $\mathbb{R}^2$; this angle is a classical counterpart to the Berry phase [Be2]. In our case, however, we have $\Delta\theta(\mathcal{C}) = 0$ for any loop $\mathcal{C}$. Indeed, since parallel translation are canonical transformation on the phase space – which follows from the fact that $H^0$ is a quadratic polynomial in coordinate and momentum coordinates – all the Hamiltonians $H(\vec{a})$ have identical canonical frequencies. If, moreover, the potential is such that $H(\vec{0})$ has a purely discrete spectrum, then the vanishing of the Hannay angle is a consequence of the Berry correspondence principle [Be2], $\Delta\theta(\mathcal{C}) = -\partial\gamma_n(\mathcal{C})/\partial n$, since in our case the Berry phase is independent of the index $n$ labeling the energy levels.

The appearance of a nontrivial adiabatic phase factor in a moving potential $V_{\vec{a}}$ is thus a purely quantum phenomenon like, e.g., the well-known Aharonov-Bohm effect. This feature distinguished the effect discussed in this letter from the Berry phase related to the Larmor precession of an electron bound to a fixed centre of attraction such as a heavy nucleus [Ham].

Let us finish with another remark concerning an extension of the above result to three-dimensional systems. Since the homogeneous field is parallel to the $z$-axis we find easily that $U_3(\vec{a}) = 0$. Consequently, the Berry phase along a closed loop $\mathcal{C}$ is again $\gamma(\mathcal{C}) = 2\pi \,\mathrm{sgn}\, e\, \Phi_C/\Phi_0$, up to a sign, where $\Phi_C$ is now the magnetic flux through the projection of $\mathcal{C}$ to a plane perpendicular to the field.


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